Vehicle Dynamics and Simulation

Using Eigenvalues and Eigenvectors

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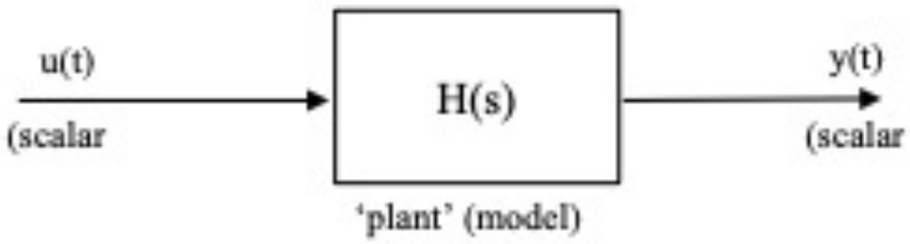
Lecture overview

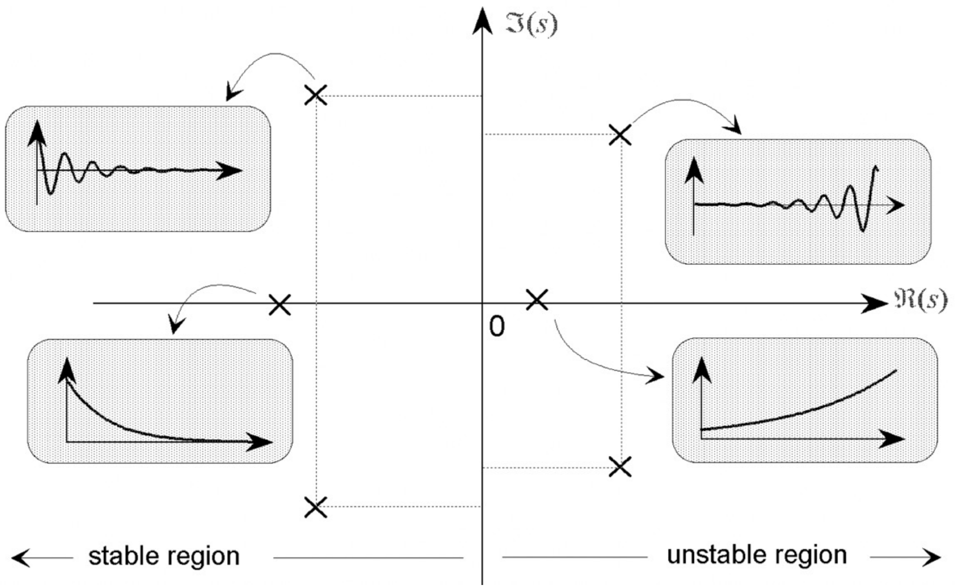
- Transfer functions
- Modal motion in free vibration
    - Eigenvalues
    - Eigenvectors

Transfer Functions

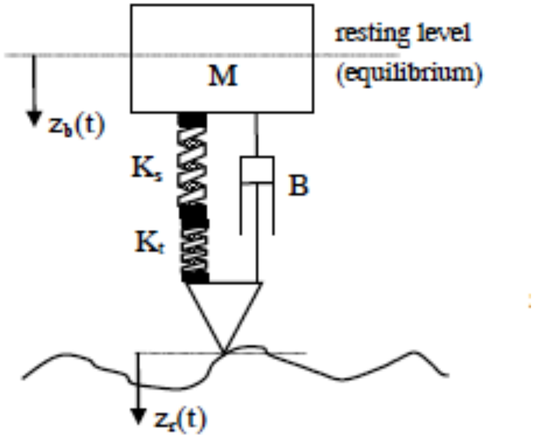
- Transfer functions relate input to output
- The roots/poles of the characteristic equation determine frequency and damping of each mode i.e. the dynamics of the system
- In state space form additional information is also available describing mode shapes from the A matrix



$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$



Simple Example



- Using the simple suspension example from Section 2b
  - Assume zero initial conditions
  - Take Laplace transform
  - Write transfer function
  - Enter parameter values
- The roots of the characteristic equation in this example are;

$$-1.875 \pm 6.372i$$

- The nature of the roots e.g. complex, repeated, distinct and real determine the general solution approach. They also define the dynamics of the system.

$$H(s) = \frac{Y(s)}{U(s)} = \frac{Bs + K}{Ms^2 + Bs + K}$$

$$H(s) = \frac{3.75s + 44.1}{s^2 + 3.75s + 44.1}$$

Laplace Transform and the Transfer Function

- State space representation;

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \qquad [1]$$

$$\boldsymbol{y} = C\boldsymbol{x} + D\boldsymbol{u} \qquad [2]$$

- Assuming zero initial conditions and taking the Laplace transform of [1];

$$sX = AX + BU$$

$$(sI - A)X = BU$$

$$X = (sI - A)^{-1}BU$$

- Substituting into [2];

$$Y = C(sI - A)^{-1}BU + DU$$

$$\boxed{H(s) = C(sI - A)^{-1}B + D} \qquad [3]$$

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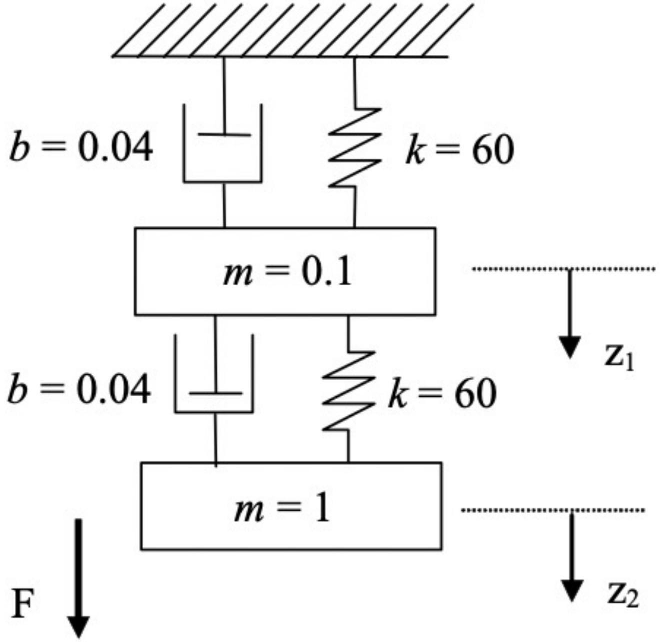
Laplace Transform and the Transfer Function

- Equation [3] provides a general solution in terms of the transfer function, $H(s)$ and is an **<u>alternate form</u>** to the State Space Representation.
- Comparing … r the eigenvalue…
- It can be sh… he Characteris…
- The eigenv…
- Matrix A go… e system will freely vibrate as it settles after some initial disturbance (**<u>not the inputs</u>**).

The eigenvalues therefore tell us about the damping and natural frequency of each mode of the system

worked example (reminder)

An Example



- The equations of the system are (expressed in terms of two second order differential equations);

$$M_2\ddot{z}_2 = F - k(z_2 - z_1) - b(\dot{z}_2 - \dot{z}_1)$$

$$M_1\ddot{z}_1 = k(z_2 - z_1) + b(\dot{z}_2 - \dot{z}_1) - kz_1 - b\dot{z}_1$$

Choosing one deflection and one velocity state per mass;

$$x_1 = z_1 \qquad x_3 = \dot{z}_1$$
$$x_2 = z_2 \qquad x_4 = \dot{z}_2$$

and the input, $u = F$. So that the set of equations describing the state derivatives becomes;

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = x_4$$

$$\dot{x}_3 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2 - \frac{2b}{m}x_3 + \frac{b}{m}x_4$$

$$\dot{x}_4 = kx_1 - kx_2 + bx_3 - bx_4 + u$$

So that the state space representation becomes;

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2k/m & k/m & -2b/m & b/m \\ k & -k & b & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \qquad [3]$$

$$\dot{\boldsymbol{X}} = \boldsymbol{AX} + \boldsymbol{BU}$$

Modal motion in free vibration – Eigenvalues

- The vector of deflections only, $\boldsymbol{z(t)} = [x_1 \;\; x_2]^T$ for our example may be written as a linear combination;

$$\boldsymbol{z(t)} = Re\left\{\boldsymbol{u}_1 e^{\lambda_1 t} + \boldsymbol{u}_2 e^{\lambda_2 t}\right\}$$

where each term $\boldsymbol{u_i} e^{\lambda_i t}$ represents a single vibrational mode, $\boldsymbol{u}_i$ are complex constants [2x1 vector in this example], λ_i are complex scalars and $\max(i) = n$ with n being the number of states.

- Evaluating a single term in the above, split λ_i into real and imaginary parts;

$$\lambda_i = \sigma + bi$$

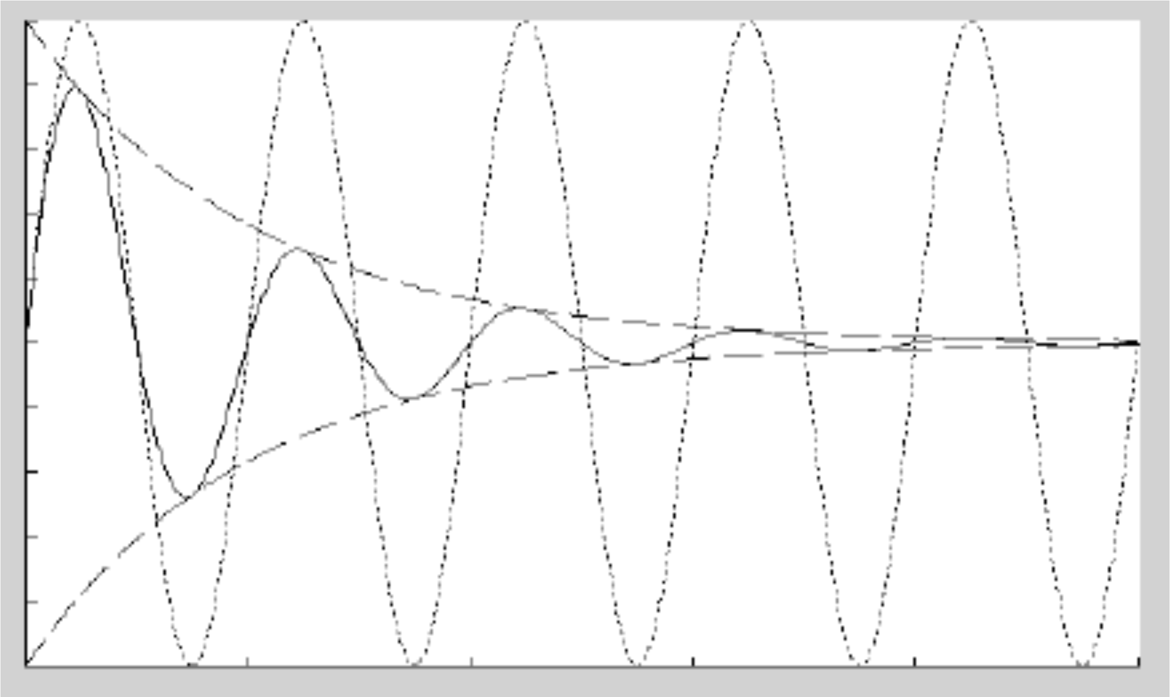
- Using the above and Euler's formula we can better evaluate what is happening;

$$\boldsymbol{u}_i e^{\lambda_i t} = \boldsymbol{u}_i e^{(\sigma + bi)t} = \boldsymbol{u}_i e^{\sigma t} e^{ibt}$$

$$\boxed{\boldsymbol{u}_i e^{\lambda_i t} = \boldsymbol{u}_i e^{\sigma t}(\cos(bt) + i\sin(bt))} \qquad [5]$$

Modal motion in free vibration - Eigenvalues

- From [5], σ should be *-ve* bounding the response to a decaying exponential, b gives the frequency of the sinusoidal component, u_i (complex) determines the magnitude and the relative phase of each mode.



$$\boldsymbol{u}_i e^{\lambda_i t} = \boldsymbol{u}_i e^{\sigma t}(\cos(bt) + i\sin(bt))$$

Modal decomposition of response
Solid line = total response
Short dash = sinusoidal component
Long dash = exponential decay

Modal motion in free vibration – Eigenvalues

- Eigenvalues appear in (complex conjugate) pairs and can be written;

$$\lambda_{1,2} = \sigma \pm j\omega_d$$

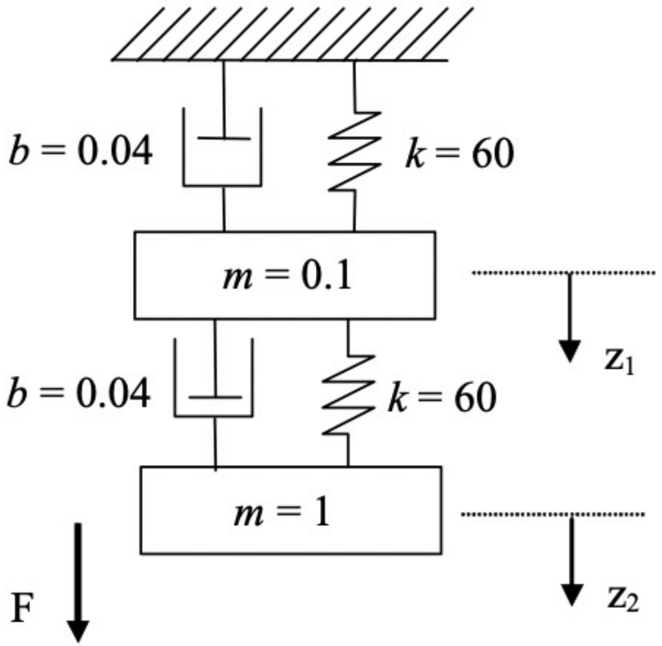
where σ is the modal damping factor and ω_d is the damped natural frequency.

Check for yourself

- Use the eig() function in MATLAB to determine the eigenvalues of matrix A from the previous example.
- How are the complex conjugate pairs placed within the resulting vector?

Check for yourself

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1200 & 600 & -0.8 & 0.4 \\ 60 & -60 & 0.04 & -0.04 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$



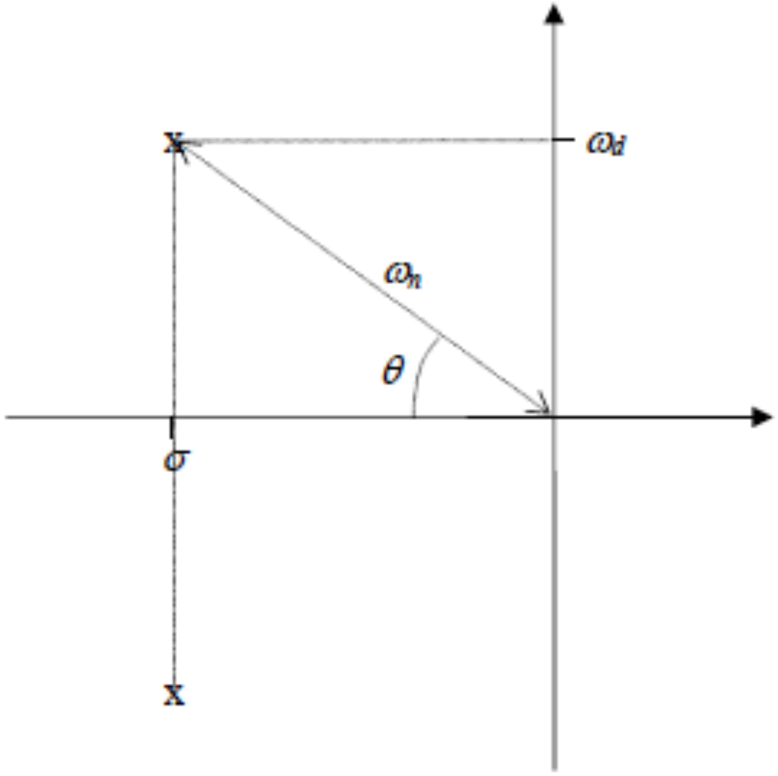
$$\lambda_{1,2} = \sigma \pm j\omega_d$$

```
D =

  -0.41025 +     35.08i          0 +         0i          0 +         0i          0 +         0i
         0 +         0i   -0.41025 -     35.08i          0 +         0i          0 +         0i
         0 +         0i          0 +         0i -0.0097502 +    5.4084i          0 +         0i
         0 +         0i          0 +         0i          0 +         0i -0.0097502 -    5.4084i
```

Modal motion in free vibration – Eigenvalues

- From the eigenvalues it is possible to tell
  - Damped natural frequency, ω_d
  - Natural frequency [Hz], $\omega_n/2\pi$
  - Damping factor, σ
  - Damping ratio, $\zeta = \cos(\theta)$
  - Settling time (within 2%), $T_s = \frac{4}{|\sigma|}$
  - Percent overshoot, $100e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$
- Note: $\lambda = 0$ corresponds to the steady-state response of the system (not dynamics)



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Check for yourself

- Using the previous example find the eigenvalues of the system and hence determine;
  - Damped natural frequency, ω_d
  - Natural frequency [Hz], $\omega_n/2\pi$
  - Damping factor, σ
  - Damping ratio, $\zeta = \cos(\theta)$
  - Settling time (within 2%), $T_s = \frac{4}{|\sigma|}$
  - Percent overshoot, $100e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$

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Modal motion in free vibration – Eigenvectors

- Eigenvectors can show the magnitudes at which the states vibrate in relation to one another.
- Writing eigenvalues and eigenvectors together in matrix form;

$$\boldsymbol{A}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{D}$$

where;

$$\boldsymbol{V} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \\ \lambda_1 u_1 & \lambda_2 u_2 & \cdots & \lambda_n u_n \end{bmatrix} \text{ and } \boldsymbol{D} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Modal motion in free vibration – Eigenvectors

- Using MATLAB ‘eig(A)’ to find the eigenvectors of the example system, A matrix;

V =

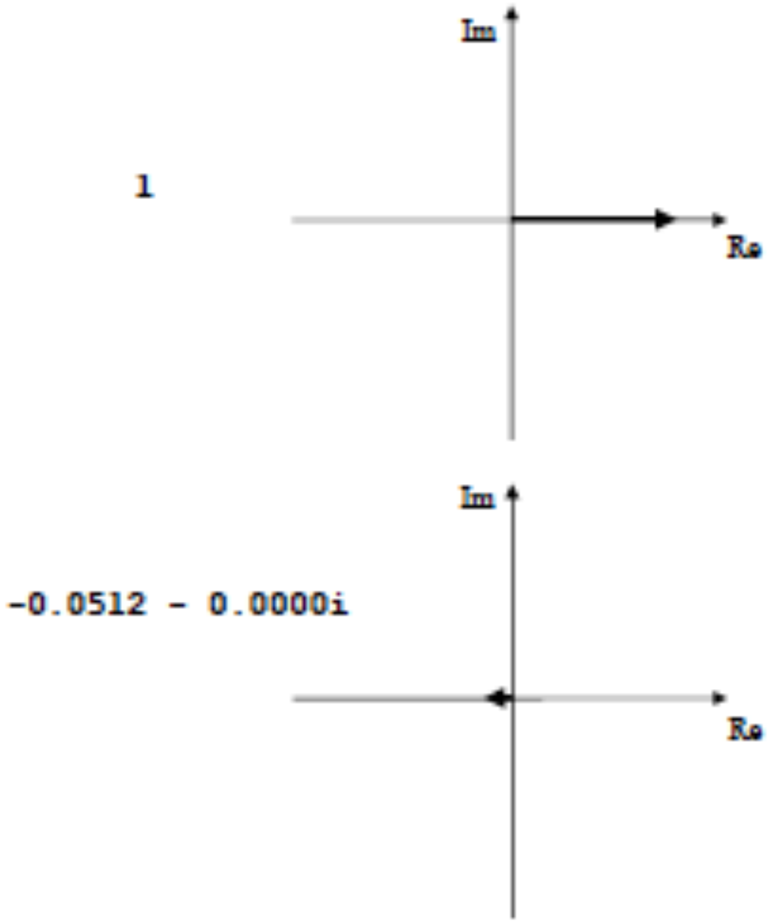
-0.0284 - 0.0017i	-0.02[illegible] + [illegible]017i	0.0665 + 0.0495i	0.06[illegible] - 0[illegible]95i
0.0015 + 0.0001i	0.001[illegible].0001i	0.1298 + 0.0966i	0.1298[illegible]0966i
0.0700 [illegible].9958i	0.070[illegible]9958i	-0.268[illegible] + [illegible].3593i	-0.2684[illegible]3593i
-0.0036 [illegible].0510i	-0.0[illegible] - [illegible]510i	-0.5236 [illegible] 0.7011i	-0.52[illegible] - 0[illegible]11i

- Note
  - The second and fourth columns are the complex conjugates of the first and third columns respectively
  - Rows three and four are the first and second rows multiplied by their respective eigenvalues
  - The system can then be characterized by considerably less ‘unique information’

Modal motion in free vibration – Eigenvectors

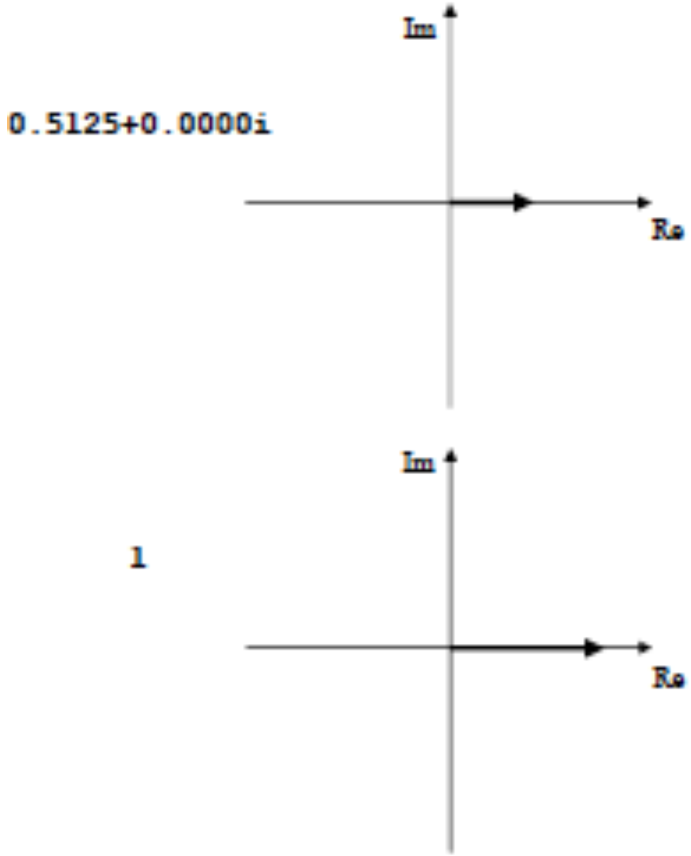
- Dividing through by the largest magnitude eigenvector $(-0.0284 - 0.017i)$ to normalize the eigenvectors.
- Plot the eigenvector components (first mode)
- The relative magnitude and phase is seen on the two plots

Mode 1



Modal motion in free vibration – Eigenvectors

- Similarly for the second (non-conjugate) mode of interest
- The relative magnitude and phase is seen on the two plots
- Note the differences between first and second modes of vibration



Mode 2

Conclusions

- Transfer function vs state space representation
- Eigenvalues tell us;
  - Damped natural frequency
  - Natural frequency
  - Damping factor
  - Damping ratio
  - Settling time
  - Percent overshoot
- Eigenvectors help us to understand vibration of the modes relative to one another